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Locally Supported Kernels for Spherical Spline Interpolation

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By the use of locally supported basis functions for spherical spline interpolation the applicability of this approximation method is extended, since the resulting interpolation matrix is sparse and thus efficient solvers can be used. In this paper we study locally supported kernels in detail. Investigations on the Legendre coefficients allow a characterization of the underlying Hilbert space structure. We show how spherical spline interpolation with polynomial precision can be managed with locally supported kernels, thus giving the possibility to combine approximation techniques based on spherical harmonic expansions with those based on locally supported kernels.

1. INTRODUCTION

Spherical splines have been successfully applied for analyzing both, exact and noisy data discretely given on the sphere, cf. e.g. [7, 9, 24, 25]. They have also been proved to be a powerful tool for the solution of boundary value problems with discretely given data, even for non–spherical boundaries (see [8, 10, 22]). The advantage of spherical spline interpolation lies mainly in two points: (i) It is applicable (under mild conditions) to all scattered data situations on the sphere. (ii) Since the spline interpolant can be proved to be the solution of a variational problem (in particular it minimizes a certain semi-norm under interpolatory constraints) unpleasant oscillations can be avoided.

Unfortunately, the progress of spherical spline interpolation was restrained, since all applied kernels had a support which covers the whole sphere, so that the resulting interpolation matrix had in general no zero entries. Thus, the areas of application were restricted to problems with not too many given data points.

This situation has completely changed when it was recognized that also kernels with a local support could be used within the existing framework, cf. [11, 19, 21]. Hence the numerical effort for storing and solving the

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linear system coming from the interpolation conditions can be drastically reduced by the use of sparse matrix solvers. Although the numerical applicability has been proved in several examples, there are some gaps in the theory of spherical spline interpolation with locally supported kernels. It is the intention of this paper to close these gaps.

Of basic importance for the characterization of the underlying Hilbert space structure are the Legendre coefficients occuring in the series expansion of the reproducing kernels. Of particular interest are those coefficients that become zero for a certain given kernel, since this means that the Hilbert space does not contain spherical harmonics of the corresponding degree. It will be shown in this paper, that for a large class of locally supported kernels the zeros of the Legendre coefficients correspond to the zeros of certain Gegenbauer polynomials. Another aspect is that it was so far not known how spherical spline interpolation with locally supported kernels can be managed with polynomial precision up to a certain order. This problem will also be solved in this paper. Finally it is shown how error estimates can be proved for the described interpolation procedure.

The advantages of the locally supported kernels are not for free. One has to cope with the calamity, that a representation of the (iterated) kernels in terms of elementary functions is not known. But this difficulty can be overcome, since the series expansion of the kernels in terms of Legendre polynomials can be used for an a-priori approximation of the kernel by e.g. a piecewise linear function. This can be done, of course, with any desired accuracy.

The paper is organized as follows: after some preliminary facts given in Chapter 2, we will briefly review the existing spherical spline theory in Chapter 3. In Chapter 4 we shall then show the main results: locally supported kernels are introduced, and their Legendre expansion is analyzed. We will investigate the zero coefficients of the Legendre expansion of the kernel to arrive at a characterization of the underlying Hilbert spaces. Finally, locally supported kernels for spline interpolation with polynomial precision are constructed. Chapter 5 shows then that the introduced kernels are applicable for spline interpolation and gives some error estimates. After that, some conclusions are drawn in the sixth chapter.

2. PRELIMINARIES

Let $\Omega = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ be the unit sphere in $\mathbb{R}^3$ with surface measure $d\omega$. Throughout this paper we use greek letters $(\xi, \eta, ...)$ to denote points of Ω. The scalar product in $\mathbb{R}^3$ is written by $\cdot$. Let $\mathscr{L}^2(\Omega)$ and $\mathscr{C}^{(p)}(\Omega)$ stand for the space of square-integrable or p-times continuously differentiable (real) functions on Ω, respectively. The usual $\mathscr{L}^2(\Omega)$-inner product is denoted by $(\cdot, \cdot)_{\mathscr{L}^2(\Omega)}$.

For the reader's convenience, we list some basic facts on the theory of spherical harmonics used in this paper. As reference, we mention e.g. [16]. Let Δ^* denote the Beltrami operator on Ω. Then it is well-known that the equation $\Delta^* F = \lambda F$ is solvable for an infinitely often differentiable $F: \Omega \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ if and only if $\lambda = -n(n+1)$ for some $n \in \mathbb{N}_0$. The infinitely often differentiable eigenfunctions of Δ^* with respect to the eigenvalue $-n(n+1)$ are the spherical harmonics of order n. We collect them in the space Harm_n. Its dimension is $\dim(\mathrm{Harm}_n) = 2n+1$. We assume in the following that $\{Y_{n,1}, ..., Y_{n,2n+1}\}$ is an $\mathscr{L}^2(\Omega)$-orthonormal system of spherical harmonics of order n. Since spherical harmonics of different order are known to be orthogonal, we can introduce the orthogonal sum $\mathrm{Harm}_{0,...,m} = \bigoplus_{n=0}^{m} \mathrm{Harm}_n$. Furthermore, we obtain the orthonormal set $\{Y_{n,j}\}_{n=0,...,\, j=1,...,2n+1}$, which is known to be complete in $\mathscr{L}^2(\Omega)$ with respect to $(\cdot,\cdot)_{\mathscr{L}^2(\Omega)}$ and closed in $\mathscr{C}(\Omega)$ with respect to the uniform topology.

Of importance for the following studies is the addition theorem, which says that for any $n \in \mathbb{N}_0$ and all $\xi, \eta \in \Omega$ it holds

$$\sum_{j=1}^{2n+1} Y_{n,j}(\xi)\, Y_{n,j}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta),$$

where $P_n: [-1, 1] \to \mathbb{R}$ is the Legendre polynomial of degree n. They are known to satisfy the recurrence relation

$$\begin{gathered} P_0(t) = 1, \qquad P_1(t) = t, \\ (n+1)\, P_{n+1}(t) + nP_{n-1}(t) - (2n+1)\, tP_n(t) = 0, \qquad n \geqslant 1. \end{gathered} \tag{1}$$

Furthermore, we have for $t \in [-1, 1]$

$$P'_{n+1}(t) - P'_{n-1}(t) = (2n+1)\, P_n(t), \qquad n \geqslant 1. \tag{2}$$

For later use, we mention the following estimate (cf. [14]) for $t \in (-1, 1)$:

$$|P_n(t)| \leqslant \sqrt{\frac{2}{\pi(n+1/2)\sqrt{1-t^2}}}. \tag{3}$$

From the addition theorem and the completeness of the spherical harmonics it follows that the Fourier expansion of $F \in \mathscr{L}^2(\Omega)$ can be written as

$$F \sim \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_{\Omega} F(\eta)\, P_n(\eta \cdot\,)\, d\omega(\eta).$$

Let $G \in \mathscr{L}^2[-1,1]$ and $\eta \in \Omega$ be fixed. The η-*zonal* function $G(\eta\,\cdot\,)\colon \Omega \to \mathbb{R}$ given by $\xi \mapsto G(\eta \cdot \xi)$ is in $\mathscr{L}^2(\Omega)$ and is axisymmetric with respect to the axis η, i.e. the value at the point $\xi \in \Omega$ depends only on the inner product $\xi \cdot \eta$. Since $|\xi - \eta| = \sqrt{2 - 2\xi \cdot \eta}$, zonal functions can be seen to be the spherical counterpart to radial basis functions in Euclidean spaces. The Funk–Hecke formula tells us that for any $Y_n \in \mathrm{Harm}_n$ and $\eta \in \Omega$,

$$\int_\Omega G(\eta \cdot \xi)\, Y_n(\xi)\, d\omega(\xi) = G^\wedge(n)\, Y_n(\eta),$$

where

$$G^\wedge(n) = 2\pi \int_{-1}^{1} G(t)\, P_n(t)\, dt.$$

Applying the addition theorem it follows that the Fourier expansion of the η-zonal function $G(\eta\,\cdot\,)$ is

$$G(\eta\,\cdot\,) \sim \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} G^\wedge(n)\, P_n(\eta\,\cdot\,). \tag{4}$$

Due to a result of [13] the series on the right hand side of (4) converges in uniform sense to $G(\eta\,\cdot\,)$, provided that G is Lipschitz–continuous on $[-1,1]$.

We use zonal functions to introduce *spherical convolutions*, see e.g. [4] or [5]. For $G \in \mathscr{L}^2[-1,1]$ and $F \in \mathscr{L}^2(\Omega)$ we set

$$(G * F)(\xi) = \int_\Omega G(\xi \cdot \eta)\, F(\eta)\, d\omega(\eta), \qquad \xi \in \Omega.$$

Of particular importance is the convolution with a second zonal function: let $H \in \mathscr{L}^2[-1,1]$. An easy application of the Funk–Hecke formula shows that

$$(G * H)(\xi, \zeta) = \int_\Omega G(\xi \cdot \eta)\, H(\eta \cdot \zeta)\, d\omega(\eta), \qquad \xi, \zeta \in \Omega,$$

depends only on the inner product of ξ and ζ, (cf. also [3]). Thus, $(G * H)(\,\cdot\,\zeta)$ is a ζ-zonal function, or in other words, $G * H$ can be seen to be a function defined on the interval $[-1,1]$. In the following we will make frequently use of this identification. It can be easily seen that for $G, H \in \mathscr{L}^2[-1,1]$

$$(G * H)^\wedge(n) = G^\wedge(n)\, H^\wedge(n), \qquad n \in \mathbb{N}_0.$$

The convolution of a function $G \in \mathscr{L}^2[-1, 1]$ with itself leads to the so-called *iterated functions* $G^{(q)}: [-1, 1] \to \mathbb{R}$ by the successive definition

$$G^{(1)}(\xi \cdot \zeta) = G(\xi \cdot \zeta),$$

$$G^{(q+1)}(\xi \cdot \zeta) = \int_\Omega G^{(q)}(\xi \cdot \eta)\, G^{(1)}(\eta \cdot \zeta)\, d\omega(\eta), \qquad q \geqslant 1.$$

Obviously, $(G^{(q)})^\wedge(n) = [G^\wedge(n)]^q$. Note that if $\operatorname{supp} G = [h, 1]$, we trivially have

$$\operatorname{supp} G(\eta \cdot\,) = \{\xi \in \Omega \mid h \leqslant \xi \cdot \eta \leqslant 1\},$$

and it is not difficult to see that

$$\operatorname{supp} G^{(2)}(\eta \cdot\,) = \begin{cases} \{\xi \in \Omega \mid 2h^2 - 1 \leqslant \xi \cdot \eta \leqslant 1\} & \text{for} \quad h > 0 \\ \Omega & \text{for} \quad -1 \leqslant h \leqslant 0. \end{cases} \tag{5}$$

3. SPHERICAL SPLINES

In this chapter we state the general theory of spherical splines as developed in [7, 9, 24, 25]. Since the theory is well-established in the meanwhile, our introduction will be brief and all proofs are omitted. However, in spite of the settings in [9], we present a slightly generalized theory, since we allow some Legendre coefficients of the reproducing kernel to be zero.

3.1. *Reproducing Kernel Hilbert Spaces*

We start with

DEFINITION 3.1. Let $\{A_n\}_{n=0,1,\ldots} \subset \mathbb{R}$ be a sequence. We introduce a decomposition $\mathbb{N}_0 = \mathscr{N} \cup \mathscr{N}_0$, $\mathscr{N} \cap \mathscr{N}_0 = \varnothing$ by

$$\mathscr{N} = \{n \in \mathbb{N}_0 \mid A_n \neq 0\}$$

$$\mathscr{N}_0 = \{n \in \mathbb{N}_0 \mid A_n = 0\}.$$

The sequence $\{A_n\}_{n=0,\ldots}$ is called τ-*summable*, $\tau \in \mathbb{R}$, if

$$\sum_{n \in \mathscr{N}} \frac{2n+1}{4\pi} \frac{n^{2\tau}}{A_n^2} < \infty.$$

A 0-summable sequence is simply called *summable*.

Let the sequence $\{A_n\}_{n=0,\dots}$ be fixed. For the space

$$\mathscr{E}(\{A_n\};\Omega)=\left\{F\in\mathscr{C}^{(\infty)}(\Omega)\mid (F,Y_{n,j})_{\mathscr{L}^2(\Omega)}=0 \text{ for all } n\in\mathscr{N}_0 \text{ and}\right.$$
$$\left.\sum_{n\in\mathscr{N}} A_n^2\sum_{j=1}^{2n+1}(F,Y_{n,j})_{\mathscr{L}^2(\Omega)}<\infty\right\}$$

we introduce an innner product by

$$(F,G)_{\mathscr{H}(\{A_n\};\Omega)}=\sum_{n\in\mathscr{N}}A_n^2\sum_{j=1}^{2n+1}(F,Y_{n,j})_{\mathscr{L}^2(\Omega)}(G,Y_{n,j})_{\mathscr{L}^2(\Omega)},$$
$$F,G\in\mathscr{E}(\{A_n\};\Omega),$$

and define the space $\mathscr{H}(\{A_n\};\Omega)$ to be the completion $\mathscr{H}(\{A_n\};\Omega)=\overline{\mathscr{E}(\{A_n\};\Omega)}$, where the completion is understood with respect to the topology given by $(\cdot,\cdot)_{\mathscr{H}(\{A_n\};\Omega)}$. Then we end up with a Hilbert space. If there is no confusion likely to arise, we write in the following $\mathscr{H}$ instead of $\mathscr{H}(\{A_n\};\Omega)$.

The order of summability of the sequence $\{A_n\}_{n=0,\dots}$ determines the smoothness of the functions in $\mathscr{H}$ in the following way:

LEMMA 3.2. *Let $\{A_n\}_{n=0,\dots}$ be τ-summable. Then $\mathscr{H}\subset\mathscr{C}^{(k)}(\Omega)$ if $\tau\geqslant k$.*

The proof of this lemma is straightforward by considering the series expansion of a function in $\mathscr{H}$ and using estimates for the derivatives of the Legendre polynomials.

The Hilbert space $\mathscr{H}$ corresponding to a summable sequence $\{A_n\}_{n=0,\dots}$ admits a reproducing kernel $K_{\mathscr{H}}(\cdot,\cdot)$ since the evaluation functionals $\mathscr{H}\ni F\mapsto F(\xi)$ are continuous for every $\xi\in\Omega$ (cf. [2]). It can be easily seen, that $K_{\mathscr{H}}(\cdot,\cdot)$ is given by the uniformly convergent series

$$K_{\mathscr{H}}(\xi,\eta)=\sum_{n\in\mathscr{N}}\frac{2n+1}{4\pi}\frac{1}{A_n^2}P_n(\xi\cdot\eta).$$

Since the function $K_{\mathscr{H}}(\cdot,\eta)$ is an η-zonal function, we frequently write $K_{\mathscr{H}}(\xi,\eta)=K_{\mathscr{H}}(\xi\cdot\eta)$, i.e. we assume $K_{\mathscr{H}}$ to be defined on the interval $[-1,1]$. Note that the Legendre coefficients are given by

$$K_{\mathscr{H}}^{\wedge}(n)=\begin{cases}A_n^{-2} & \text{for } n\in\mathscr{N}\\ 0 & \text{for } n\in\mathscr{N}_0\end{cases}.$$

Since they satisfy $K_{\mathscr{H}}^{\wedge}(n)\geqslant 0$, the kernels $K_{\mathscr{H}}$ are positive definite, as we shall see in the next section.

Let $m\in\mathbb{N}_0$ and assume that the summable sequence $\{A_n\}_{n=0,\dots}$ satisfies $A_n\neq 0$ for $n=0,\dots,m$. Then the space $\mathscr{H}=\mathscr{H}(\{A_n\};\Omega)$ can be decomposed into the orthogonal sum $\mathscr{H}=\mathscr{H}_{0,\dots,m}\oplus\mathscr{H}^{\perp}_{0,\dots,m}$, where $\mathscr{H}_{0,\dots,m}=\mathrm{Harm}_{0,\dots,m}$. Both subspaces are then reproducing kernel Hilbert spaces with respect to $(\cdot,\cdot)_{\mathscr{H}}=(\cdot,\cdot)_{\mathscr{H}_{0,\dots,m}}+(\cdot,\cdot)_{\mathscr{H}^{\perp}_{0,\dots,m}}$. Obviously, the corresponding reproducing kernels are

$$K_{\mathscr{H}_{0,\dots,m}}(\xi,\eta)=\sum_{n=0}^{m}\frac{2n+1}{4\pi}\frac{1}{A_n^2}P_n(\xi\cdot\eta)$$

$$K_{\mathscr{H}^{\perp}_{0,\dots,m}}(\xi,\eta)=\sum_{\substack{n\in\mathscr{N}\\ n>m}}\frac{2n+1}{4\pi}\frac{1}{A_n^2}P_n(\xi\cdot\eta),$$

i.e. $K_{\mathscr{H}}(\cdot,\cdot)=K_{\mathscr{H}_{0,\dots,m}}(\cdot,\cdot)+K_{\mathscr{H}^{\perp}_{0,\dots,m}}(\cdot,\cdot)$. The norm $\|\cdot\|_{\mathscr{H}^{\perp}_{0,\dots,m}}$ of $\mathscr{H}^{\perp}_{0,\dots,m}$ can be seen to be a seminorm in $\mathscr{H}$ with kernel $\mathrm{Harm}_{0,\dots,m}$. This fact will be important for the forthcoming definition of spherical splines.

3.2. *Definition of Spherical Splines*

The basic idea of spherical spline interpolation can be described as follows: assume that there are given a set of nodal points $X_N=\{\eta_1,\dots,\eta_N\}\subset\Omega$ and a set of values $\{y_1,\dots,y_N\}\subset\mathbb{R}$. Suppose further that there is a fixed reproducing kernel space $\mathscr{H}$ which is orthogonally decomposed into $\mathscr{H}=\mathscr{H}_{0,\dots,m}\oplus\mathscr{H}^{\perp}_{0,\dots,m}$, as described in the last section. Then the task of spline interpolation is to find a solution $S\in\mathscr{H}$ of

$$\|S\|_{\mathscr{H}^{\perp}_{0,\dots,m}}=\min_{F\in\mathscr{I}_N}\|F\|_{\mathscr{H}^{\perp}_{0,\dots,m}},\tag{6}$$

where $\mathscr{I}_N=\{F\in\mathscr{H}\mid F(\eta_i)=y_i,\ i=1,\dots,N\}$. In other words, we search for the interpolant $S\in\mathscr{H}$, which has the smallest $\mathscr{H}^{\perp}_{0,\dots,m}$-norm.

It follows from (6) that if the data come from a function in the space $\mathrm{Harm}_{0,\dots,m}$, this function is exactly reproduced by the spline. That is, what we call *polynomial precision* of order m. Note that we also treat the case formally written by $m=-1$. In this situation the Hilbert space $\mathscr{H}$ is not decomposed and the spline interpolation minimizes the norm $\|S\|_{\mathscr{H}}$ among all interpolating functions.

In order to ensure the well-posedness of problem (6), we have to make some assumption on the set X_N of nodal points and on the reproducing kernel $K_{\mathscr{H}}$ (in full analogy to spline theory in Euclidean spaces, cf. e.g. [15]). Let for $m\in\mathbb{N}_0$, $M=\sum_{n=0}^{m}(2n+1)=(m+1)^2$ be the dimension of $\mathrm{Harm}_{0,\dots,m}$. We call a pointset $X_M=\{\eta_1,\dots,\eta_M\}$ *fundamental system with respect to* $\mathrm{Harm}_{0,\dots,m}$, if the matrix

$$\begin{pmatrix} Y_{0,1}(\eta_1) & \cdots & Y_{m,2m+1}(\eta_1)\\ \vdots & \ddots & \vdots\\ Y_{0,1}(\eta_M) & \cdots & Y_{m,2m+1}(\eta_M)\end{pmatrix}$$

is regular. In this case, the problem of interpolation in $\mathrm{Harm}_{0, \ldots, m}$ with respect to X_M is uniquely solvable. Let $N \geqslant M$. Then we call a pointset $X_N = \{\eta_1, \ldots, \eta_N\} \subset \Omega$ of pairwise different points *admissible* with respect to $\mathrm{Harm}_{0, \ldots, m}$, if it contains a fundamental system with respect to $\mathrm{Harm}_{0, \ldots, m}$. By changing the order we can always assume that the set $\{\eta_1, \ldots, \eta_M\} \subset X_N$ is unisolvent with respect to $\mathrm{Harm}_{0, \ldots, m}$. We will see lateron that this is exactly the requirement on X_N to ensure that the spline interpolation problem is uniquely solvable.

However, also the reproducing kernel has to satisfy a condition:

DEFINITION 3.3. A function $K: [-1, 1] \to \mathbb{R}$ is called *strictly positive definite*, if

$$\sum_{i=1}^{N} \sum_{k=1}^{N} a_i K(\eta_i \cdot \eta_k)\, a_k > 0 \tag{7}$$

for all choices of pointsets $X_N = \{\eta_1, \ldots, \eta_N\}$ with pairwise distinct points, and all non-zero vectors $(a_1, \ldots, a_N) \in \mathbb{R}^N$. If equality with zero is also allowed in (7), the function K is called *positive definite*.

Positive definite kernels for radial basis approximation in Euclidean spaces have been investigated in [15], where also the strict positive definiteness of a large class of radial basis functions has been shown. (For a general overwiew on the theory of radial basis functions in Euclidean spaces, see e.g. [17].) In the spherical case, Schoenberg [20] has proved, that K is positive definite if and only if $K^\wedge(n) \geqslant 0$ for all $n \in \mathbb{N}_0$. A sufficient condition for K to be strictly positive definite is found in [26], see also [11]. However, this condition is not applicable for the locally supported kernels, as we shall see later.

For spline interpolation with polynomial precision a weaker condition on the kernels can be applied:

DEFINITION 3.4. The continuous function $K: [-1, 1] \to \mathbb{R}$ is called *conditionally stricly positive definite of order m*, if (7) holds for any $\mathrm{Harm}_{0, \ldots, m}$-admissible system $X_N = \{\eta_1, \ldots, \eta_N\}$ and any $(a_1, \ldots, a_N) \neq 0$ satisfying

$$\begin{pmatrix} Y_{0,1}(\eta_1) & \cdots & Y_{0,1}(\eta_N) \\ \vdots & \ddots & \vdots \\ Y_{m,2m+1}(\eta_1) & \cdots & Y_{m,2m+1}(\eta_N) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = 0.$$

Let $\{A_n\}_{n=0,\ldots}$ be a summable sequence. Assume $m \in \mathbb{N}_0$ to be fixed and consider the space $\mathcal{H} = \mathcal{H}(\{A_n\}; \Omega)$ with orthogonal decomposition $\mathcal{H} = \mathcal{H}_{0, \ldots, m} \oplus \mathcal{H}^{\perp}_{0, \ldots, m}$. Let $X_N = \{\eta_1, \ldots, \eta_N\} \subset \Omega$ be an admissible system with respect to $\mathrm{Harm}_{0,\ldots,m}$ and assume that the reproducing kernel $K_{\mathcal{H}^{\perp}_{0,\ldots,m}}$ is conditionally strictly positive definite of order m. Then we call any function $S \in \mathcal{H}$ of the form

$$S(\xi) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} Y_{n,j}(\xi) + \sum_{i=1}^{N} a_i K_{\mathcal{H}^{\perp}_{0,\ldots,m}}(\eta_i, \xi), \qquad \xi \in \Omega, \tag{8}$$

spherical spline with respect to X_N, if the coefficients $a_1, \ldots, a_N$ satisfy the linear equations

$$\sum_{i=1}^{N} a_i Y_{n,j}(\eta_i) = 0, \qquad n = 0, \ldots, m, \quad j = 1, \ldots, 2n+1.$$

Note that here and in the following the results hold also for the special case, where we do not consider the orthogonal decomposition of $\mathcal{H}$, but perform the interpolation directly in $\mathcal{H}$. This case was denoted before by $m = -1$.

We summarize the main results on spherical spline interpolation in the following theorem:

THEOREM 3.5. *Let* m, $\mathcal{H} = \mathcal{H}_{0,\ldots,m} \oplus \mathcal{H}^{\perp}_{0,\ldots,m}$ *and* X_N *be as before. Assume that there is a given data vector* $(y_1, \ldots, y_N) \in \mathbb{R}^N$. *Then there exists a uniquely defined solution in* $S \in \mathcal{H}$ *of the spline interpolation problem*

$$\|S\|_{\mathcal{H}^{\perp}_{0,\ldots,m}} = \min_{F \in \mathcal{I}_N} \|F\|_{\mathcal{H}^{\perp}_{0,\ldots,m}}$$

where $\mathcal{I}_N = \{F \in \mathcal{H} \mid F(\eta_i) = y_i,\ i = 1, \ldots, N\}$. S *is a spherical spline of the form* (8), *where the coefficients* $c_{n,j}$ *and* a_i *solve the system of linear equations*

$$\sum_{i=1}^{N} a_i K_{\mathcal{H}^{\perp}_{0,\ldots,m}}(\eta_k, \eta_i) + \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} Y_{n,j}(\eta_k), \qquad k = 1, \ldots, N,$$

and

$$\sum_{i=1}^{N} a_i Y_{n,j}(\eta_i) = 0, \qquad n = 0, \ldots, m, \quad j = 1, \ldots, 2n+1.$$

Furthermore, any $F \in \mathcal{I}_N$ *satisfies*

$$\|F\|^2_{\mathcal{H}^{\perp}_{0,\ldots,m}} = \|S\|^2_{\mathcal{H}^{\perp}_{0,\ldots,m}} + \|S - F\|^2_{\mathcal{H}^{\perp}_{0,\ldots,m}}.$$

For the proof of this theorem and many results on the proper choice of the space $\mathcal{H}$, i.e. the choice of the kernel $K_{\mathcal{H}}$ we recommend [9] or [11].

It is obvious that the system of linear equations stated in the last theorem is very labourous to solve, since all the matrix elements are in general different from zero. However, if the kernel $K_{\mathscr{H}^{\perp}_{0,\ldots,m}}$ (respectively $K_{\mathscr{H}}$ for the case $m=-1$) had only a local support, the matrix would be sparse, and so the system could be solved very efficiently. Therefore, we will show in the following how splines with locally supported kernels can be constructed.

4. LOCALLY SUPPORTED KERNELS

In this chapter, we introduce locally supported kernels that allow the definition of reproducing kernel Hilbert spaces and which will be proved to be strictly positive definite, so that they can be applied for spline interpolation. We will consider both types of kernels, namely $K_{\mathscr{H}}$ and $K_{\mathscr{H}^{\perp}_{0,\ldots,m}}$, the second one being characterized by $K^{\wedge}_{\mathscr{H}^{\perp}_{0,\ldots,m}}(n)=0$ for $n=0, \ldots, m$. The basic idea for the construction of the kernels is to start with a simple locally supported one-dimensional function and then to use the spherical convolution in order to assure that all the Legendre coefficients are non-negative. The construction of the kernels $K_{\mathscr{H}^{\perp}_{0,\ldots,m}}$ uses then (besides the convolution) certain linear combinations of simpler kernels. For the characterization of the underlying space $\mathscr{H}$, it is important to know, which Legendre coefficients of the constructed kernels are zero. This will be discussed in detail in Section 4.2.

4.1. *Definitions and Recurrence Relations*

We start with the definition of the piecewise polynomial function $B_h^{(k)}: [-1, 1] \to \mathbb{R}$, for $k=0, 1, \ldots$, and $h \in (0, 1)$, which has already been considered in [12] in another context. We set

$$B_h^{(k)}(t)=\begin{cases} 0 & \text{for} \quad -1 \leqslant t \leqslant h \\ (t-h)^k/(1-h)^k & \text{for} \quad h<t \leqslant 1 \end{cases}. \tag{9}$$

Let $\eta \in \Omega$ be fixed. Then the η-zonal function $B_h^{(k)}(\eta\cdot): \Omega \to \mathbb{R}$ has the local support

$$\operatorname{supp} B_h^{(k)}(\eta\cdot)=\{\xi \in \Omega \mid h \leqslant \xi\cdot\eta \leqslant 1\}.$$

From (5) we know that the iterated η-zonal function $(B_h^{(k)})^{(2)}(\eta\cdot)$ has the support $\operatorname{supp}(B_h^{(k)})^{(2)}(\eta\cdot)=\{\xi \in \Omega \mid 2h^2-1 \leqslant \xi\cdot\eta \leqslant 1\}$. It is also obvious, that if we allowed $h \leqslant 0$, we would get $\operatorname{supp}(B_h^{(k)})^{(2)}(\eta\cdot)=\Omega$, i.e. the function would be supported over the whole sphere Ω. That is the reason why we restrict the parameter h to the interval $(0, 1)$.

Expanding $B_h^{(k)}$ in terms of Legendre polynomials, we find

$$B_h^{(k)} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} B_h^{(k)\wedge}(n) P_n,$$

where by the Funk–Hecke formula the Legendre coefficients $B_h^{(k)\wedge}(n)$ are given by

$$B_h^{(k)\wedge}(n) = 2\pi \int_h^1 \frac{(t-h)^k}{(1-h)^k} P_n(t)\, dt.$$

A recursion formula for the $B_h^{(k)\wedge}(n)$ can be derived by considering the integrals

$$I_{n,k} = \int_h^1 (t-h)^k P_n(t)\, dt. \tag{10}$$

Straightforward integration yields

$$I_{0,k} = \frac{(1-h)^{k+1}}{k+1}, \qquad I_{1,k} = I_{0,k} \frac{k+1+h}{k+2}.$$

From the recurrence formula (1) we see $(n \geqslant 1)$

$$(n+1) I_{n+1,k} + nI_{n-1,k} - (2n+1) I_{n,k+1} - (2n+1) hI_{n,k} = 0.$$

Furthermore, we find by partial integration and (2) that

$$(2n+1) I_{n,k+1} = -(k+1)(I_{n+1,k} - I_{n-1,k}). \tag{11}$$

Combining these results, we arrive at the following recursion formula:

$$(n+k+2) I_{n+1,k} = (2n+1) hI_{n,k} + (k+1-n) I_{n-1,k}.$$

This gives us

LEMMA 4.1. *For $k \geqslant 0$ and $n \geqslant 1$ we have*

$$B_h^{(k)\wedge}(0) = 2\pi \frac{1-h}{k+1}$$

$$(k+2) B_1^{(k)\wedge}(1) = (k+1+h) B_h^{(k)\wedge}(0)$$

$$(n+k+2) B_h^{(k)\wedge}(n+1) = (2n+1) hB_h^{(k)\wedge}(n) + (k+1-n) B_h^{(k)\wedge}(n-1).$$

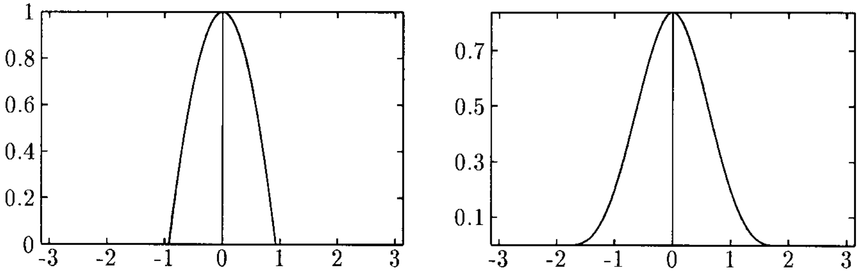

FIG. 1. The kernels $B_h^{(1)}(\cos \vartheta)$ (left) and the iterated kernel $(B_h^{(1)})^{(2)}(\cos \vartheta)$ (right) for the parameter $h=0.6$.

Next we shall consider the asymptotic behaviour of $|B_h^{(k)\wedge}(n)|$ for fixed $h \in (0, 1)$ as $n \to \infty$. Let $h \in (0, 1)$ be fixed. For $k=0$ and $n \geqslant 1$ we get from (2)

$$I_{n,0} = \int_h^1 P_n(t)\, dt = \frac{1}{2n+1} (P_{n+1}(h) - P_{n-1}(h)).$$

Using estimate (3) we see that $|B_h^{(0)\wedge}(n)| = O(n^{-3/2})$ as $n \to \infty$. From (11) we conclude recursively that $|B_h^{(k)\wedge}(n)| = O(n^{-3/2-k})$, as n tends to infinity. Thus, for $k \geqslant 0$,

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (B_h^{(k)})^{(2)\wedge}(n) < \infty,$$

i.e. the sequence $[B_h^{(k)\wedge}(n)]^{-1}$, $n=0, \ldots$, is summable. Therefore the kernel $(B_h^{(k)})^{(2)}$ can be seen to be the reproducing kernel of the space $\mathscr{H} = \mathscr{H}(\{1/B_h^{(k)\wedge}(n)\}; \Omega)$ (cf. Figure 1). Note, that it may happen that $B_h^{(k)\wedge}(n) = 0$ for some $n \in \mathbb{N}$. In this case, $1/B_h^{(k)\wedge}(n)$ has then to be substituted by zero in the definition of the space $\mathscr{H}$. Nevertheless, for a complete characterization of $\mathscr{H}$ it is important to know which numbers $B_h^{(k)\wedge}(n)$ are zero. This is the topic of the next section.

4.2. *Zeros in the Legendre Expansion*

For a complete characterization of the space $\mathscr{H} = \mathscr{H}(\{1/B_h^{(k)\wedge}(n)\}; \Omega)$ we have to answer the question which coefficients in the Legendre expansion of the kernel $B_h^{(k)}$ are zero. We will show now, that this answer is related to the zeros of certain Gegenbauer polynomials.

The main result of this section is

THEOREM 4.2. *For* $h \in (0, 1)$ *and* $k = 0, 1, \ldots$, *we have*:

(i) $B_h^{(k)\wedge}(n) \neq 0$ *for* $n = 0, 1, \ldots, k+2$.

(ii) *For* $n \geqslant k+2$, $B_h^{(k)\wedge}(n)=0$ *if and only if* $C_{n-k-1}^{k+3/2}(h)=0$, *where* $C_m^{k+3/2}$ *denotes the Gegenbauer polynomial of order m with respect to* $\lambda = k+3/2$.

Before we come to the proof of this theorem, we mention the following consequence:

Corollary 4.3. *Let* $k \geqslant 0$.

(i) *There exists* $h \in (0, 1)$ *such that* $B_h^{(k)\wedge}(n) \neq 0$ *for all* $n \in \mathbb{N}_0$.

(ii) *Let* $m \in \mathbb{N}_0$ *be given. Then there exists a number* $h_0 \in (0, 1)$ *such that for all* $h \in (h_0, 1)$ *and all* $n \leqslant m$, $B_h^{(k)\wedge}(n) \neq 0$.

Proof. Part (i) follows immediately from Theorem 4.2 since there exist only countably many points being a zero of a Gegenbauer polynomial.

It follows from the theory of orthogonal polynomials that the largest number $h \in (0, 1)$ for which $C_{n-k-1}^{k+3/2}(h)=0$ for a n with $k+2 \leqslant n \leqslant m$ is the largest zero of $C_{m-k-1}^{k+3/2}$ (cf. e.g. [23]). So just take this value as h_0. This proves part (ii). ∎

The first result is of more theoretical concern, since it is also known that the set of all zeros of the Gegenbauer polynomials $C_n^{k+3/2}$, $n=0, ...,$ is dense in $[-1, 1]$. But the second statement is very useful. It shows that it is possible by suitable choices of the parameter $h \in (0, 1)$ to ensure that all spherical harmonics up to a certain order are contained in the space $\mathscr{H}$.

Proof of Theorem 4.2. We know from Lemma 4.1 that $B_h^{(k)\wedge}(0) > 0$ and $B_h^{(k)\wedge}(1) > 0$. Furthermore it follows from the recurrence formula that $B_h^{(k)\wedge}(n+1) > 0$ as long as $n \leqslant k+1$. This proves (i).

The second statement is a consequence of the results shown below in this section. ∎

Lemma 4.1 shows that each $B_h^{(k)\wedge}(n)$ is a polynomial of degree $n+1$ in the variable h which is zero for $h=1$. Thus, the linear factor $(1-h)$ can be divided, and we define

$$b_n^{(k)}(h) = \frac{1}{2\pi(1-h)} B_h^{(k)\wedge}(n), \qquad n=0, 1, ..., h \in (0, 1).$$

Then $b_n^{(k)}$ is a polynomial of degree n in h which we will consider for $h \in [-1, 1]$. From the recurrence relation for the $B_h^{(k)\wedge}(n)$ (cf. Lemma 4.1) we immediately get the following recursion formula for $b_n^{(k)}(h)$ $(n \geqslant 1)$:

$$\begin{aligned} b_0^{(k)}(h) &= \frac{1}{k+1} \\ (k+2)\, b_1^{(k)}(h) &= (k+1+h)\, b_0^{(k)}(h) \\ (n+k+2)\, b_{n+1}^{(k)}(h) &= (2n+1)\, h b_n^{(k)}(h) + (k+1-n)\, b_{n-1}^{(k)}(h). \end{aligned} \tag{12}$$

As first result we mention

LEMMA 4.4. *For all $k \geqslant 0$ we have*

$$b_0^{(k)}(-1) = \frac{1}{k+1}$$

$$b_n^{(k)}(-1) = \frac{(k+1-n)\cdot \cdots \cdot (k-1)k}{(k+1)\cdot \cdots \cdot (k+1+n)}, \qquad n \geqslant 1.$$

Proof. Immediately by induction using (12). ∎

LEMMA 4.5. *The lth derivatives, $l \geqslant 0$, of $b_n^{(k)}$ satisfy for $h = -1$:*

(i) *for $l > n$:* $\quad (b_n^{(k)})^{(l)}(-1) = 0$

(ii) *for $l = n$:* $\quad (b_n^{(k)})^{(l)}(-1) = \dfrac{(2l)!}{2^l} \dfrac{1}{(k+1)\cdot \cdots \cdot (k+1+l)}$

(iii) *for $l < n$:*

$$(b_n^{(k)})^{(l)}(-1) = \frac{(n-l+1)\cdot \cdots \cdot (n+l)}{2^l} \frac{(k+1-n)\cdot \cdots \cdot (k-l)}{(k+1)\cdot \cdots \cdot (k+1+n)}.$$

Proof. Since $b_n^{(k)}$ is a polynomial of degree n, part (i) is obvious. We prove part (ii) and (iii) by induction. For $l = 0$ the assertion is just Lemma 4.4. An easy calculation shows also the validity for $l = 1$. Assume now, (ii) and (iii) are true for all $l' \leqslant l$. Then we obtain from the recursion formula and Leibniz' rule

$$\begin{aligned}(b_{l+1}^{(k)})^{(l+1)}(h) &= \frac{2l+1}{l+k+2} h (b_l^{(k)})^{(l+1)}(h) + (l+1)\frac{2l+1}{l+k+2}(b_l^{(k)})^{(l)}(h) \\ &\quad + \frac{k+1-l}{l+k+2}(b_{l-1}^{(k)})^{(l)}(h) \\ &= (l+1)\frac{2l+1}{l+k+2}(b_l^{(k)})^{(l)}(h).\end{aligned}$$

Part (ii) follows then immediately by setting $h = -1$.

In order to prove the result for $(b_n^{(k)})^{(l+1)}(-1)$ if $n \geqslant l+1$ we now argue again by a nested induction, this time with respect to the variable n. For $n = l+1$ the result has just been proved. A similar calculation yields the case $n=l+2$. Under the assumption that (iii) is true for all n' with $l+1 \leqslant n' \leqslant n$ we have

$$(b_{n+1}^{(k)})^{(l+1)}(h) = \frac{2n+1}{n+k+2} h(b_n^{(k)})^{(l+1)}(h) + (l+1)\frac{2n+1}{n+k+2}(b_n^{(k)})^{(l)}(h) + \frac{k+1-n}{n+k+2}(b_{n-1}^{(k)})^{(l+1)}(h).$$

Setting $h=-1$ and inserting the known formulas for $(b_n^{(k)})^{(l+1)}$, $(b_n^{(k)})^{(l)}$ and $(b_{n-1}^{(k)})^{(l+1)}$ the result follows after some lengthy but straightforward calculations. ∎

This lemma shows in particular that $(b_{k+1}^{(k)})^{(l)}(h)=0$ at the point $h=-1$ for $l=0, ..., k$. But since $b_{k+1}^{(k)}$ is a polynomial of degree $k+1$ it is clear that $b_{k+1}^{(k)}$ is of the form

$$b_{k+1}^{(k)} = \alpha_k(h+1)^{k+1} \tag{13}$$

with a constant α_k. This constant can be determined by taking the $(k+1)$st derivative in (13) and substituting $h=-1$. With the results of Lemma 4.5 we arrive at

$$\frac{(2k+2)!}{2^{k+1}} \frac{1}{(k+1)\cdot \cdots \cdot (2k+2)} = \alpha_k(k+1)!.$$

Thus, $\alpha_k = 1/(2^{k+1}(k+1))$.

Using the recursion formula (12) we see $b_{k+2}^{(k)}(h) = hb_{k+1}^{(k)}(h)$. As a consequence, all $b_n^{(k)}(h)$, $n \geqslant k+1$ have a zero of order $k+1$ at $h=-1$. Hence the definition

$$c_n^{(k)}(h) = \frac{1}{\alpha_k(h+1)^{k+1}} b_{n+k+1}^{(k)}(h), \qquad n \geqslant 0, \tag{14}$$

yields a polynomial in h of degree n. Note that for $n \geqslant k+1$ it holds $B_h^{(k)\wedge}(n)=0$ if and only if $c_{n-k-1}^{(k)}(h)=0$. Hence, Theorem 4.2 is proved, if we have shown that $c_n^{(k)}$ is up to a multiplicative constant the Gegenbauer polynomial C_n^λ with $\lambda = k+3/2$. This is shown in the last lemma of this section:

Lemma 4.6. *For $k \geqslant 0$, $n \in \mathbb{N}_0$, and $\lambda = k+3/2$*

$$C_0^\lambda(h) = c_0^{(k)}(h)$$

$$C_n^{(\lambda)}(h) = \frac{1}{n!} \prod_{j=0}^{n-1} (2k+3+j)\, c_n^{(k)}(h), \qquad n \geqslant 1.$$

Proof. Set

$$\tilde{C}_0^\lambda(h) = c_0^{(k)}(h)$$

$$\tilde{C}_n^\lambda(h) = \frac{1}{n!} \prod_{j=0}^{n-1} (2k+3+j)\, c_n^{(k)}(h), \qquad n \geqslant 1.$$

We will show that the $\tilde{C}_n^\lambda$ satisfy the recurrence relation of the Gegenbauer polynomials

$$\begin{gathered} C_0^\lambda(h) = 1 \\ C_1^\lambda(h) = 2\lambda h \\ (n+1)\, C_{n+1}^\lambda(h) - 2(\lambda+n)\, hC_n^\lambda(h) + (2\lambda+n-1)\, C_{n-1}^\lambda(h) = 0, \qquad n \geqslant 1, \end{gathered} \tag{15}$$

for $\lambda = k + 3/2$, cf. e.g. [1] or [23].

It is obvious that the initial values fulfil

$$\tilde{C}_0^\lambda(h) = c_0^{(k)}(h) = \frac{1}{\alpha_k}\, b_{k+1}^{(k)}(h) = 1$$

and

$$\tilde{C}_1^\lambda(h) = (2k+3)\, c_1^{(k)}(h) = (2k+3)\, \frac{1}{\alpha_k (h+1)^{k+1}}\, b_{k+2}^{(k)}(h)$$

$$= (2k+3)\, h\, \frac{1}{\alpha_k (h+1)^{k+1}}\, b_{k+1}^{(k)}(h) = (2k+3) h = 2\lambda h.$$

Using (12) and (14) we obtain from

$$(n+2k+3)\, b_{n+k+2}^{(k)}(h) - (2n+2k+3)\, hb_{n+k+1}^{(k)}(h) + nb_{n+k}^{(k)}(h) = 0$$

that

$$(n+2k+3)\, \tilde{C}_{n+1}^\lambda(h) - \frac{(2k+3+n)(2n+2k+3)}{n+1}\, h\tilde{C}_n^\lambda(h)$$

$$+ \frac{(2k+3+n)(2k+2+n)n}{n(n+1)}\, \tilde{C}_{n-1}^\lambda(h) = 0.$$

Thus,

$$(n+1)\, \tilde{C}_{n+1}^\lambda(h) - 2(\lambda+n)\, \tilde{C}_n^\lambda(h) + (2\lambda+n-1)\, \tilde{C}_{n-1}^\lambda(h) = 0,$$

which is just (15), as required. ∎

4.3. *Reproducing Kernel Spaces with Locally Supported Kernels*

Summarizing our previous results we have proved the following

THEOREM 4.7. *Let for $k \geqslant 0$ and $h \in (0,1)$ the function $B_h^{(k)}: [-1,1] \to \mathbb{R}$ be defined as in (9). Let the sets $\mathcal{N} = \{n \in \mathbb{N}_0 \mid B_h^{(k)\wedge}(n) \neq 0\}$ and $\mathcal{N}_0 = \{n \in \mathbb{N}_0 \mid B_h^{(k)\wedge}(n) = 0\}$ be determined by Theorem 4.2. The sequence*

$$A_n = \begin{cases} 1/B_h^{(k)\wedge}(n) & \text{for} \quad n \in \mathcal{N} \\ 0 & \text{for} \quad n \in \mathcal{N}_0 \end{cases} \tag{16}$$

is summable and defines the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\{A_n\}; \Omega)$. The reproducing kernel of $\mathcal{H}$ is then

$$K_{\mathcal{H}}(\xi, \eta) = (B_h^{(k)})^{(2)}(\xi \cdot \eta), \qquad \xi, \eta \in \Omega,$$

with

$$\operatorname{supp} K_{\mathcal{H}}(\ ,\eta) = \{\xi \in \Omega \mid 2h^2 - 1 \leqslant \xi \cdot \eta \leqslant 1\}.$$

Thus, reproducing kernel Hilbert spaces with locally supported reproducing kernels are found. If we have shown that the kernels $K_{\mathcal{H}} = (B_h^{(k)})^{(2)}$ are strictly positive definite functions, the spline interpolation can be performed efficiently for the case $m = -1$, i.e. without polynomial precision. However, taking $m \geqslant 0$, the modified kernels $K_{\mathcal{H}^{\perp}_{0,\dots,m}}$ arising in the interpolation matrix, are now globally supported. Thus, we shall now present a method to construct also kernels $K_{\mathcal{H}^{\perp}_{0,\dots,m}}$ that are locally supported.

The idea is simple: let $m \in \mathbb{N}_0$ be fixed. Choose $m+2$ pairwise different $h_1 < \cdots < h_{m+2} \in (0,1)$. Then determine real numbers $k_1, \dots, k_{m+2}$ in such a way that

$$K(t) = \sum_{i=1}^{m+2} k_i B_{h_i}^{(k)}(t), \qquad t \in [-1,1]$$

satisfies

$$K^{\wedge}(n) = 0, \qquad n = 0, \dots, m. \tag{17}$$

If we then define

$$K_{\mathcal{H}}(\xi, \eta) = \sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta) + K^{(2)}(\xi \cdot \eta), \qquad \xi, \eta \in \Omega, \tag{18}$$

we obtain the reproducing kernel of the space $\mathscr{H}=\mathscr{H}(\{A_n\};\Omega)$, where the sequence $\{A_n\}_{n=0,\ldots}$ is given by

$$A_n=\begin{cases}1/(k_1 B_{h_1}^{(k)\wedge}(n)+\cdots+k_{m+2}B_{h_{m+2}}^{(k)\wedge}(n)) & \text{for}\quad n\in\mathscr{N}\\ 1 & \text{for}\quad n=0,\ldots,m.\\ 0 & \text{for}\quad n\in\mathscr{N}_0\end{cases}$$

Thereby we have used $\mathscr{N}=\{n\in\mathbb{N}_0 \mid \sum_{i=1}^{m+2} k_i B_{h_i}^{(k)\wedge}(n)\neq 0\}$ and $\mathscr{N}_0=\{n\geqslant m+1 \mid \sum_{i=1}^{m+2} k_i B_{h_i}^{(k)\wedge}(n)=0\}$. Furthermore, it follows that $K_{\mathscr{H}_{0,\ldots,m}^{\perp}}=K^{(2)}$, and hence

$$\operatorname{supp} K_{\mathscr{H}_{0,\ldots,m}^{\perp}}(\eta\cdot)=\{\xi\in\Omega \mid 2h_1^2-1\leqslant\xi\cdot\eta\leqslant 1\}$$

for all $\eta\in\Omega$. So it remains to find $k_1,\ldots,k_{m+2}\in\mathbb{R}$ so that (17) is fulfilled.

THEOREM 4.8. *Let for fixed $m\in\mathbb{N}_0$, $0<h_1<\cdots<h_{m+2}<1$. Then there exist $k_1,\ldots,k_{m+2}\in\mathbb{R}$ so that*

$$K(t)=\sum_{i=1}^{m+2} k_i B_{h_i}^{(k)}(t),\qquad t\in[-1,1],$$

satisfies

$$K^{\wedge}(n)=0\qquad \textit{for}\quad n=0,\ldots,m. \tag{19}$$

If we require in addition

$$k_1+\cdots+k_{m+2}=1, \tag{20}$$

the numbers k_i are uniquely determined. The support of $K^{(2)}(\eta\cdot)$ is for fixed $\eta\in\Omega$ given by

$$\operatorname{supp} K^{(2)}(\eta\cdot)=\{\xi\in\Omega \mid 2h_1^2-1\leqslant\xi\cdot\eta\leqslant 1\}.$$

Proof. We only have to show that the system of linear equations defined by (19) and (20) is uniquely solvable. It can be written as

$$\begin{pmatrix}1 & \cdots & 1\\ B_{h_1}^{(k)\wedge}(0) & \cdots & B_{h_{m+2}}^{(k)\wedge}(0)\\ \vdots & \ddots & \vdots\\ B_{h_1}^{(k)\wedge}(m) & \cdots & B_{h_{m+2}}^{(k)\wedge}(m)\end{pmatrix}\begin{pmatrix}k_1\\ k_2\\ \vdots\\ k_{m+2}\end{pmatrix}=\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}. \tag{21}$$

But we have seen in Lemma 4.1 that the system $\{1, B_h^{(k)\wedge}(0),\ldots,B_h^{(k)\wedge}(m)\}$ is a system of polynomials in h with degrees $0,\ldots,m+1$, respectively. Thus,

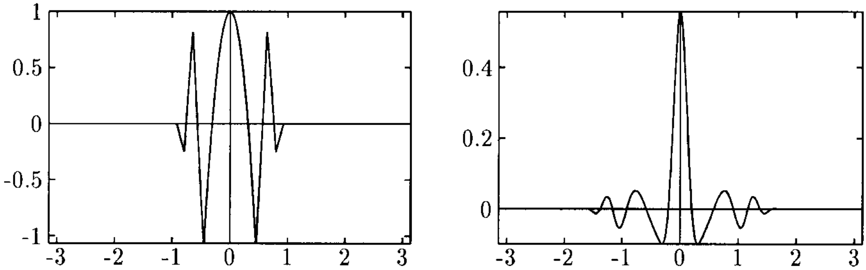

FIG. 2. The kernel $K(\cos \vartheta)$ of Theorem 4.8 for $m=2$ (left). The values $h_1, ..., h_4$ and $k_1, ..., k_4$ are given by $(h_1, ..., h_4) = (0.6, 0.7, 0.8, 0.9)$, $(k_1, ..., k_4) = (-1, 4, -6, 4)$. On the right hand side there is the iterated kernel $K^{(2)}(\cos \vartheta)$.

they build a unisolvent system for interpolation on the interval $[-1, 1]$ (cf. [6]). Therefore the system (21) is uniquely solvable, as required. The property of the support of K is obvious from the previous discussion. ∎

An example is illustrated in Figure 2.

5. SPHERICAL SPLINES WITH LOCALLY SUPPORTED BASIS FUNCTIONS

In order to establish spherical splines based on locally supported kernels one aspect is missing, namely the conditionally strict positive definiteness of the described kernels, which will ensure the regularity of the linear system to be solved for spline interpolation. This is the topic of Section 5.1.

The estimations in uniform norm of the spline interpolant as proved e.g. in [9] are given by the use of series expansions in terms of the Legendre polynomials. For our situation, error estimates can be obtained in an easier way, as shown in Section 5.2.

5.1. *Strict Positive Definiteness*

A sufficient condition for the spline interpolation matrix to be solvable is the conditionally strict positive definiteness of order m of the reproducing kernel $K_{\mathscr{H}^{\perp}_{0, ..., m}}$. We are able to show even more:

THEOREM 5.1. *Let* $k \geqslant 0$.

(i) *If* $h \in (0, 1)$ *then* $(B_h^{(k)})^{(2)}$ *is strictly positive definite.*

(ii) *Let* $m \in \mathbb{N}_0$ *and* $0 < h_1 < \cdots < h_{m+2} < 1$, *and assume that* $K: [-1, 1] \to \mathbb{R}$ *is defined as in Theorem* 4.8. *Then* $K^{(2)}$ *is strictly positive definite.*

Proof. Assertion (i) is equivalent to the positive definiteness of the symmetric matrix

$$\begin{pmatrix} (B_h^{(k)})^{(2)}(\eta_1\cdot\eta_1) & \cdots & (B_h^{(k)})^{(2)}(\eta_1\cdot\eta_N) \\ \vdots & \ddots & \vdots \\ (B_h^{(k)})^{(2)}(\eta_N\cdot\eta_1) & \cdots & (B_h^{(k)})^{(2)}(\eta_N\cdot\eta_N) \end{pmatrix} \tag{22}$$

for any pointset $X_N=\{\eta_1, ..., \eta_N\}$ of pairwise distinct points. However, since

$$\begin{aligned}(B_h^{(k)})^{(2)}(\eta_i\cdot\eta_k) &= \int_\Omega B_h^{(k)}(\eta_i\cdot\zeta)\, B_h^{(k)}(\zeta\cdot\eta_k)\, d\omega(\zeta) \\ &= (B_h^{(k)}(\eta_i\cdot\,),\, B_h^{(k)}(\eta_k\cdot\,))_{\mathscr{L}^2(\Omega)},\end{aligned}$$

the matrix (22) can be seen to be a Gram matrix, and is therefore positive definite if and only if the functions $B_h^{(k)}(\eta_i\cdot\,)$, $i=1, ..., N$, are linearly independent. But this can be concluded from the discontinuity of the kth derivative of $B_h^{(k)}(\cdot)$ in the following way: assume that $F=\sum_{i=1}^N a_i B_h^{(k)}(\eta_i\cdot\,)=0$. Thus, F is trivially of class $\mathscr{C}^{(\infty)}$. On the other hand side, the kth derivative of a function $\xi\mapsto B_h^{(k)}(\eta_i\cdot\xi)$ in direction $\eta_i-(\eta_i\cdot\xi)\xi$ does not exist for all $\xi\in\Omega_i$ with $\Omega_i=\{\xi\in\Omega \mid \xi\cdot\eta_i=h\}$. Suppose now, that an a_i (without loss of generality, say a_1) is different from zero. Since there exists a point $\xi_1\in\Omega_1\setminus\bigcup_{j=2}^N\Omega_j$, it follows that the described kth derivative of F at the point ξ does not exist, a contradiction. Thus, $a_1=\cdots=a_N=0$ is proved, and therefore part (i) is shown. The second statement follows by similar arguments. ∎

5.2. *Convergence*

An estimation in uniform topology is given by

THEOREM 5.2. *Let $h\in(0,1)$, $k\geqslant 1$, and let $\{A_n\}_{n=0,...}$ be defined as in* (16). *Then there exists a constant $C>0$ with the following property: let $F\in\mathscr{H}=\mathscr{H}(\{A_n\};\Omega)$ and assume that $X_N=\{\eta_1, ..., \eta_N\}\subset\Omega$ consists of pairwise distinct points. If $S\in\mathscr{H}$ is the uniquely determined interpolating spline with respect to the data set $\{F(\eta_1), ..., F(\eta_N)\}$ then we have for all $\xi\in\Omega$*

$$|S(\xi)-F(\xi)|\leqslant C\,\|F\|_{\mathscr{H}}\,\Theta_N,$$

where the nodal width Θ_N is defined by

$$\Theta_N=\max_{\xi\in\Omega}\ \min_{i=1,...,N}|\xi-\eta_i|.$$

Proof. Let $\xi \in \Omega$. Then there exists an $i \in \{1, ..., N\}$ such that $|\xi - \eta_i| \leqslant \Theta_N$. Since $S(\eta_i) = F(\eta_i)$, we obtain by the triangle inequality

$$|S_N(\xi) - F(\xi)| \leqslant |S_N(\xi) - S_N(\eta_i)| + |F(\xi) - F(\eta_i)|.$$

From the reproducing property it follows that

$$|S(\xi) - S(\eta_i)| = |(K_{\mathscr{H}}(\xi, \cdot) - K_{\mathscr{H}}(\eta_i, \cdot), S_N)_{\mathscr{H}}|$$

and

$$|F(\xi) - F(\eta_i)| = |(K_{\mathscr{H}}(\xi, \cdot) - K_{\mathscr{H}}(\eta_i, \cdot), F)_{\mathscr{H}}|.$$

By the Cauchy–Schwarz inequality we obtain

$$|S(\xi) - S(\eta_i)| \leqslant \|K_{\mathscr{H}}(\xi, \cdot) - K_{\mathscr{H}}(\eta_i, \cdot)\|_{\mathscr{H}} \ \|S\|_{\mathscr{H}}$$

$$|F(\xi) - F(\eta_i)| \leqslant \|K_{\mathscr{H}}(\xi, \cdot) - K_{\mathscr{H}}(\eta_i, \cdot)\|_{\mathscr{H}} \ \|F\|_{\mathscr{H}}.$$

Since we know from Theorem 3.5 that $\|S\|_{\mathscr{H}} \leqslant \|F\|_{\mathscr{H}}$,

$$|S(\xi) - F(\xi)| \leqslant 2 \ \|K_{\mathscr{H}}(\xi, \cdot) - K_{\mathscr{H}}(\eta_i, \cdot)\|_{\mathscr{H}} \ \|F\|_{\mathscr{H}}.$$

For the estimation of $\|K_{\mathscr{H}}(\xi, \cdot) - K_{\mathscr{H}}(\eta_i, \cdot)\|_{\mathscr{H}}$ we mention first that the function $(B_h^{(k)})^{(2)}\colon [-1, 1] \to \mathbb{R}$ is Lipschitz-continuous, i.e. for a constant $\tilde{C} > 0$ it holds $|(B_h^{(k)})^{(2)}(t) - (B_h^{(k)})^{(2)}(t')| \leqslant \tilde{C}\,|t - t'|$ for $-1 \leqslant t,\ t' \leqslant 1$. Hence,

$$\begin{aligned}
\|K_{\mathscr{H}}(\xi, \cdot) - K_{\mathscr{H}}(\eta_i, \cdot)\|_{\mathscr{H}}^2 &= |K_{\mathscr{H}}(\xi, \xi) + K_{\mathscr{H}}(\eta_i, \eta_i) - 2K_{\mathscr{H}}(\xi, \eta_i)| \\
&= 2\ |(B_h^{(k)})^{(2)}(1) - (B_h^{(k)})^{(2)}(\xi \cdot \eta_i)| \\
&\leqslant 2\tilde{C}\,|1 - \xi \cdot \eta_i| \\
&= \tilde{C}\,|\xi - \eta_i|^2 \\
&\leqslant \tilde{C}\Theta_N^2.
\end{aligned}$$

Taking the square root, the assertion follows with $C = 2\sqrt{\tilde{C}}$. ∎

The case $m \geqslant 0$, i.e. spline interpolation with polynomial precision up to order m, can be handled similarly. Since for the proof one has only to combine the methods of [9] and the arguments of the last proof, we just state

THEOREM 5.3. *Let $m \geqslant 0$ and $k \geqslant 1$. Assume that $0 < h_1 < \cdots < h_{m+2} < 1$ and let $k_1 \cdots k_{m+2} \in \mathbb{R}$ be determined as in Theorem* 4.8. *The function*

$$K_{\mathscr{H}}(t) = \left(\sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(t) + \sum_{i=1}^{m+2} k_i B_{h_i}^{(k)}(t) \right)^{(2)}, \qquad t \in [-1, 1],$$

defines the reproducing kernel of the space $\mathscr{H}$. *Let* $F \in \mathscr{H}$ *and assume* $X_N = \{\eta_1, ..., \eta_N\}$ *to be admissible with respect to* $\mathrm{Harm}_{0,...,m}$. *Assume that* $S \in \mathscr{H}$ *is the uniquely determined solution of*

$$\|S\|_{\mathscr{H}} = \min_{G \in \mathscr{I}_N} \|G\|_{\mathscr{H}_{0,...,m}^{\perp}}$$

with $\mathscr{I}_N = \{G \in \mathscr{H} \mid G(\eta_i) = F(\eta_i),\ i = 1, ..., N\}$, *as described in Theorem* 3.5. *Then there exists a constant* $C > 0$ (*only dependent on* $\mathscr{H}$ *and the unisolvent set* $\{\eta_1, ..., \eta_M\} \subset X_N$) *such that for all* $\xi \in \Omega$ *the estimate*

$$|S(\xi) - F(\xi)| \leqslant C \|F\|_{\mathscr{H}_{0,...,m}^{\perp}} \Theta_N$$

is valid, where Θ_N *is defined as in the last theorem.*

6. CONCLUSIONS

The results of the last chapters have shown how spherical spline interpolation can be performed by the use of locally supported basis functions for both types of interpolation, viz. with and without polynomial precision. Thus, the possible area of application for spherical splines is extended, since now the numerical effort is reduced dramatically. Examples for applications in relevant problems can be found in [19] or [21].

The locally supported kernels $K_{\mathscr{H}_{0,...,m}^{\perp}}$ will offer the possibility to combine the well-established methods using expansions in terms of spherical harmonics with localizing methods, as e.g. the described spline methods, but also wavelet expansions with these kernels are possible. It is assumed, that such approaches will be of importance for future representations of e.g. the earth's gravitational field, cf. e.g [18, 21].

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